

# The Functional Integration and the Two-Point Correlation Function of the One-Dimensional Bose Gas in the Harmonic Potential

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## Abstract

A quantum field-theoretical model which describes spatially non-homogeneous one-dimensional non-relativistic repulsive Bose gas in an external harmonic potential is considered. We calculate the two-point thermal correlation function of the Bose gas in the framework of the functional integration approach. The calculations are done in the coordinate representation. A method of successive integration over the “high-energy” functional variables first and then over the “low-energy” ones is used. The effective action functional for the low-energy variables is calculated in one loop approximation. The functional integral representation for the correlation function is obtained in terms of the low-energy variables, and is estimated by means of the stationary phase approximation. The asymptotics of the correlation function is studied in the limit when the temperature is going to zero while the volume occupied by non-homogeneous Bose gas infinitely increases. It is demonstrated that the behaviour of the thermal correlation function in the limit described is power-like, and it is governed by the critical exponent which depends on the spatial and thermal arguments.

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# §1. Introduction

A recent burst of interest to theory of the Bose gas is caused by experimental realization of Bose condensation in the ultra-cold vapors of alkali metals confined in the magneto-optical traps [1, 2]. In particular, it became possible to study the Bose condensation in the systems, which are effectively two-dimensional or quasi one-dimensional [3, 4]. Here a partial localization along one or two directions in three-dimensional system is achieved by making the level spacing of the trapping potential in the corresponding directions larger than the energies of individual atoms. The field models which describe the Bose particles with the delta-like inter-particle coupling confined by an external harmonic potential provide a good approximation for a theoretical approach to the experimental situation [5]. The theory of non-ideal Bose gas attracts traditionally not only physicists but also mathematicians [6–8]. For a translationally invariant homogeneous Bose gas, the field model in question corresponds to a quantum nonlinear Schrödinger equation, which admits an exact solution in the one-dimensional case [9, 10]. This fact allows to obtain closed expressions for the correlation functions [11, 12].

In real physical systems the interest to a transition from three-dimensional to one-dimensional behaviour is caused by the fact that an effective density of atoms in the one-dimensional Bose gas can be either high or low depending on the parameters of the system [13]. Here a low density implies a strong coupling between the particles [14, 15], while a high density corresponds to a weak interaction.

The present paper continues the series of the papers [16–20] devoted to investigation of the correlation functions of the Bose gas with weak repulsive inter-particle coupling in the presence of an external harmonic potential. Since there is no exact solutions in the case of an external potential, in the present paper, as well as in Refs. [16–20], the method of functional integration [21–24] is accepted for investigation of the correlation functions. In the present paper we develop the results of the paper [20], where, in a distinction with [16–19], an explicit dependence of the correlation functions on the imaginary time for non-zero temperature was taken into account. It will be demonstrated that the presence of the external potential (of the *trap*) results in a change of the asymptotical behaviour of the two-point correlation function in comparison to a translationally invariant case. It will be demonstrated that the observed change happens in the range of temperatures, which are comparable with inverse of the characteristic length of the trap provided this length tends to infinity.

The paper is organized as follows. Section §1 has an introductory character. A description of the one-dimensional model of non-relativistic Bose field in question, as well as a summary of the method of the functional integration, are given in Section §2. An approach to approximate investigation of the functional integrals is also presented in Section §2. This approach is based on a successive integration first over the high-energy over-condensate excitations and then over the variables, which correspond to the low-energy quasi-particles. Besides, in this Section we give a derivation of one loop effective action for the low excited quasi-condensate fields, and the corresponding energy spectrum of the low lying excitations is obtained. The method of stationary phase is used in §3 for an estimation of the functional integral, which expresses the two-point thermal correlation function of the non-homogeneous Bose gas. The method of asymptotical estimation of the correlators, which is used in the present paper, was proposed in [25], where the asymptotical behaviour of two-point Green functions of the homogeneous Bose gas was investigated for the spatial dimensionalities one, two, and three. In the present paper it is demonstrated that the stationary phase method [25] admits a generalization for the spatially non-homogeneous Bose gas in the external potential

also. The asymptotics of two-point correlation functions of non-homogeneous Bose gas are obtained in Section §4 both at nonzero and zero temperatures. A short discussion of the results of the present paper is given in Section §5.

## §2. The effective action and the Thomas–Fermi approximation

### 1. The partition function

In the present paper we shall consider one-dimensional Bose gas described by the Hamiltonian  $\widehat{H}$  defined on the real axis  $\mathbb{R} \ni x$ :

$$\begin{aligned}\widehat{H} &= \int \left\{ \widehat{\psi}^\dagger(x) \mathcal{H} \widehat{\psi}(x) + \frac{g}{2} \widehat{\psi}^\dagger(x) \widehat{\psi}^\dagger(x) \widehat{\psi}(x) \widehat{\psi}(x) \right\} dx, \\ \mathcal{H} &\equiv \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - \mu + V(x),\end{aligned}\tag{1}$$

where  $\widehat{\psi}^\dagger(x)$  and  $\widehat{\psi}(x)$  are operator-valued fields, which describe creation and annihilation of the quasi-particles over the Fock vacuum  $|0\rangle$ . The fields  $\widehat{\psi}^\dagger(x)$  and  $\widehat{\psi}(x)$  are subjected to the commutation relation

$$\widehat{\psi}(x) \widehat{\psi}^\dagger(x') - \widehat{\psi}^\dagger(x') \widehat{\psi}(x) = \delta(x - x')$$

( $\widehat{\psi}^\dagger(x)$  and  $\widehat{\psi}(x)$  are mutually commutative), and  $\mathcal{H}$  is the “single-particle” Hamiltonian. The following notations are used in equation (1):  $m$  is the mass of the Bose particles,  $\mu$  is the chemical potential,  $g$  is the coupling constant which corresponds to the weak repulsion (i. e.,  $g > 0$ ), and the external confining potential is chosen in the form of the harmonic potential  $V(x) \equiv \frac{m}{2} \Omega^2 x^2$ .

Let us begin with the investigation of the partition function  $Z$ . It can be represented in the form of the functional integral [21–24]:

$$Z = \int e^{S[\psi, \bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi},\tag{2}$$

where  $S[\psi, \bar{\psi}]$  is the action functional of the system in question:

$$\begin{aligned}S[\psi, \bar{\psi}] &= \int_0^\beta d\tau \int dx \left( \bar{\psi}(x, \tau) \left( \frac{\partial}{\partial \tau} - \mathcal{H} \right) \psi(x, \tau) \right. \\ &\quad \left. - \frac{g}{2} \bar{\psi}(x, \tau) \bar{\psi}(x, \tau) \psi(x, \tau) \psi(x, \tau) \right).\end{aligned}\tag{3}$$

The domain of the functional integration in (2) is given by the space of complex-valued functions  $\bar{\psi}(x, \tau)$ ,  $\psi(x, \tau)$  depending on two arguments:  $x \in \mathbb{R}$  and  $\tau \in [0, \beta]$ . With regard to the first argument  $x$ , the functions  $\bar{\psi}(x, \tau)$ ,  $\psi(x, \tau)$  belong to the space of quadratically integrable functions  $L_2(\mathbb{R})$ , while they are finite and periodic with the period  $\beta = (k_B T)^{-1}$  with regard to the imaginary time  $\tau$  ( $k_B$  is the Boltzmann constant, and  $T$  is an absolute temperature). The variables  $\bar{\psi}$ ,  $\psi$  are the independent variables of the functional integration [21], and  $\mathcal{D}\psi \mathcal{D}\bar{\psi}$  is the functional integration measure.

At sufficiently low temperatures we can expect to assume that each of the variables  $\bar{\psi}(x, \tau)$ ,  $\psi(x, \tau)$  is given by two constituents. One of them,  $\bar{\psi}_o(x, \tau)$ ,  $\psi_o(x, \tau)$ , corresponds to *quasi-condensate*, while another one — to the high-energy thermal (i. e., over-condensate) excitations  $\bar{\psi}_e(x, \tau)$ ,  $\psi_e(x, \tau)$ :

$$\psi(x, \tau) = \psi_o(x, \tau) + \psi_e(x, \tau), \quad \bar{\psi}(x, \tau) = \bar{\psi}_o(x, \tau) + \bar{\psi}_e(x, \tau). \quad (4)$$

It should be stressed that here and below just the quasi-condensate is assumed, since a true Bose condensate does not exist in one-dimensional system [6]. In the exactly solvable case, the existence of the quasi-condensate implies that a non-trivial vacuum state (i.e., a ground state) exists. The quasi-condensate variables  $\bar{\psi}_o(x, \tau)$ ,  $\psi_o(x, \tau)$  can also be represented as the sums of the constituents:

$$\psi_o(x, \tau) = \psi_o(x) + \xi(x, \tau), \quad \bar{\psi}_o(x, \tau) = \bar{\psi}_o(x) + \bar{\xi}(x, \tau), \quad (5)$$

where the field  $\psi_o(x)$  describes the ground state of the model at zero temperature, while the field  $\xi(x, \tau)$  describes the low lying excited particles. Let us require the variables in (4) to be orthogonal in the following sense:

$$\int \psi_o(x, \tau) \bar{\psi}_e(x, \tau) dx = \int \bar{\psi}_o(x, \tau) \psi_e(x, \tau) dx = 0.$$

As a result, the integration measure  $\mathcal{D}\psi \mathcal{D}\bar{\psi}$  will be replaced by  $\mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o \mathcal{D}\psi_e \mathcal{D}\bar{\psi}_e$ .

To investigate the functional integral (2), we shall perform a successive integration over the fields  $\bar{\psi}, \psi$ . First, we shall integrate over the high-energy constituents, and then — over the low-energy ones (see. (4)) [21, 24]. At a second step, it is preferable to pass to new functional variables, which describe an observable “low-energy” physics [21, 24] in a more adequate way. After the substitution of the expansion (4) into the action (3) we shall take into account in  $S$  only the terms up to quadratic in  $\bar{\psi}_e, \psi_e$ . This means that we are making an approximation in which the over-condensate quasi-particles do not couple with each other. In this case, it is possible to integrate over the thermal fluctuations  $\bar{\psi}_e(x, \tau)$ ,  $\psi_e(x, \tau)$  in a closed form and thus to arrive to an effective action functional  $S_{\text{eff}}[\psi_o, \bar{\psi}_o]$  depending only on the quasi-condensate variables  $\psi_o, \bar{\psi}_o$ :

$$S_{\text{eff}}[\psi_o, \bar{\psi}_o] = \ln \int e^{\tilde{S}[\psi_o + \psi_e, \bar{\psi}_o + \bar{\psi}_e]} \mathcal{D}\psi_e \mathcal{D}\bar{\psi}_e, \quad (6)$$

where the tilde in  $\tilde{S}$  implies that “self-coupling” of the fields  $\bar{\psi}_e(x, \tau)$ ,  $\psi_e(x, \tau)$  is excluded. With respect to equation (6), the partition function  $Z$  of the model takes an approximate form:

$$Z \approx \int e^{S_{\text{eff}}[\psi_o, \bar{\psi}_o]} \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o. \quad (7)$$

Let us consider the derivation of the effective action  $S_{\text{eff}}[\psi_o, \bar{\psi}_o]$  (6) in more details. The splitting (4) allows to derive  $S_{\text{eff}}[\psi_o, \bar{\psi}_o]$  in the framework of the field-theoretical approach of the loop expansion [26, 27]. We substitute (4) into the initial action  $S[\psi, \bar{\psi}]$  (3) and then pass from  $S$  to the action  $\tilde{S}$ , which is given by three terms:

$$\tilde{S} = S_{\text{cond}} + S_{\text{free}} + S_{\text{int}}. \quad (8)$$

In (8),  $S_{\text{cond}}$  is the action functional of the condensate quasi-particles, which corresponds to a tree approximation [26, 27]:

$$S_{\text{cond}}[\psi_o, \bar{\psi}_o] \equiv \int_0^\beta d\tau \int dx \left\{ \bar{\psi}_o(x, \tau) \hat{K}_+ \psi_o(x, \tau) - \frac{g}{2} \bar{\psi}_o(x, \tau) \bar{\psi}_o(x, \tau) \psi_o(x, \tau) \psi_o(x, \tau) \right\}. \quad (9)$$

At the chosen approximation, the action for the over-condensate excitations  $S_{\text{free}}$  takes the form:

$$S_{\text{free}}[\psi_e, \bar{\psi}_e] \equiv \frac{1}{2} \int_0^\beta d\tau \int dx (\bar{\psi}_e, \psi_e) \hat{G}^{-1} \begin{pmatrix} \psi_e \\ \bar{\psi}_e \end{pmatrix}. \quad (10)$$

Eventually,  $S_{\text{int}}$  is given by the part of the total action functional, which describes the coupling of the quasi-condensate to the over-condensate excitations:

$$S_{\text{int}}[\psi_o, \bar{\psi}_o, \psi_e, \bar{\psi}_e] \equiv \int_0^\beta d\tau \int dx \left\{ \bar{\psi}_e(x, \tau) [\hat{K}_+ - g \bar{\psi}_o \psi_o] \psi_o(x, \tau) + \psi_e(x, \tau) [\hat{K}_- - g \bar{\psi}_o \psi_o] \bar{\psi}_o(x, \tau) \right\}. \quad (11)$$

In formulas (9)–(11), we have defined the differential operators  $\hat{K}_\pm \equiv \pm \partial/\partial\tau - \mathcal{H}$  (here the Hamiltonian  $\mathcal{H}$  is defined in (1)) and the matrix-differential operator  $\hat{G}^{-1}$ :

$$\hat{G}^{-1} \equiv \hat{G}_0^{-1} - \hat{\Sigma}, \quad (12)$$

where

$$\hat{G}_0^{-1} \equiv \begin{pmatrix} \hat{K}_+ & 0 \\ 0 & \hat{K}_- \end{pmatrix}, \quad \hat{\Sigma} \equiv \hat{\Sigma}[\psi_o, \bar{\psi}_o] = g \begin{pmatrix} 2\bar{\psi}_o \psi_o & \psi_o^2 \\ (\psi_o)^2 & 2\bar{\psi}_o \psi_o \end{pmatrix}.$$

In this approach it is appropriate to apply the stationary phase method to the functional integral (6). To this end, let us choose  $\bar{\psi}_o, \psi_o$  as the stationarity points of the functional  $S_{\text{cond}}$  (9), which are defined by the extremum condition  $\delta(S_{\text{cond}}[\psi_o, \bar{\psi}_o]) = 0$ . The corresponding equations are taking the form of the Gross–Pitaevskii-type equations [28]:

$$\begin{aligned} \left( \frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \mu - V(x) \right) \psi_o - g(\bar{\psi}_o \psi_o) \psi_o &= 0, \\ \left( -\frac{\partial}{\partial\tau} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \mu - V(x) \right) \bar{\psi}_o - g(\bar{\psi}_o \psi_o) \bar{\psi}_o &= 0. \end{aligned} \quad (13)$$

The contribution of the action  $S_{\text{int}}$  (11) drops out from (8) since  $\bar{\psi}_o, \psi_o$  are chosen to be solutions of equations (13). Therefore the dynamics of  $\psi_e, \bar{\psi}_e$  is described, in the leading approximation, by the action  $S_{\text{free}}$  (10). The latter depends on  $\bar{\psi}_o, \psi_o$  non-trivially through the matrix of the self-energy parts  $\hat{\Sigma}$ , which enters into  $\hat{G}^{-1}$  (12). The *Thomas–Fermi approximation* is essentially used in the present paper in order to determine the stationarity points  $\bar{\psi}_o, \psi_o$ . This approximation consists in neglection of the kinetic term  $\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2}$  in equations (13) [5, 28]. The Thomas–Fermi approximation is valid for the systems containing a sufficiently large number of particles, and it is widely used in the theoretical approaches

to description of the Bose condensation in the magneto-optical traps [5, 28]. The following condensate solution can be obtained provided only  $\tau$ -independent solutions of (13) are allowed:

$$\bar{\psi}_o \psi_o = \rho_{TF}(x; \mu) \equiv \frac{1}{g} (\mu - V(x)) \Theta(\mu - V(x)), \quad (14)$$

where  $\Theta$  is the Heavyside function. Now the integration in (6) with respect to  $\psi_e$ ,  $\bar{\psi}_e$  is Gaussian. This leads [21] to the one loop effective action, which takes the following form in terms of the variables  $\psi_o$ ,  $\bar{\psi}_o$ :

$$S_{\text{eff}}[\psi_o, \bar{\psi}_o] \equiv S_{\text{cond}}[\psi_o, \bar{\psi}_o] - \frac{1}{2} \ln \text{Det}(\hat{G}^{-1}). \quad (15)$$

Here  $\hat{G}^{-1}$  is the matrix operator (12), and  $\psi_o$ ,  $\bar{\psi}_o$  have a sense of the new variables, and their dynamics is governed by the action (15).

In order to assign a meaning to the final expression for the effective action (15), it is necessary to regularize the determinant  $\text{Det}(\hat{G}^{-1})$ . In our case, the operator  $\hat{G}^{-1}$  is already written as a  $2 \times 2$ -matrix Dyson equation (12), where the entries of  $\hat{\Sigma}[\psi_o, \bar{\psi}_o]$  play the role of the normal ( $\Sigma_{11} = \Sigma_{22}$ ) and anomalous ( $\Sigma_{12}$ ,  $\Sigma_{21}$ ) self-energy parts. The Dyson equation (12) defines the matrix  $\hat{G}$ , where the entries have the meaning of the Green functions of the fields  $\bar{\psi}_e, \psi_e$ . The matrix  $\hat{G}$  arises as a formal inverse of the operator  $\hat{G}^{-1}$ :

$$\hat{G} = (\hat{G}_0^{-1} - \hat{\Sigma})^{-1}. \quad (16)$$

The matrix operator  $\hat{G}^{-1}$  (12) can formally be diagonalized by means of the famous N.N.Bogoliubov's  $(u, v)$ -transform [6]. The corresponding equations, which describe the unknown coefficient-functions  $u$ ,  $v$ , result in a compatibility requirement, which determines, in turn, the quasi-classical spectrum of the elementary excitations [28].

With regard to our purposes, it is appropriate to represent  $\hat{G}^{-1}$  as follows:

$$\hat{G}^{-1} = \hat{G}_0^{-1} - \hat{\Sigma} \equiv \hat{\mathcal{G}}^{-1} - (\hat{\Sigma} - 2g\rho_{TF}(x; \mu)\hat{I}), \quad (17)$$

where  $\hat{I}$  is the unit matrix of the size  $2 \times 2$ , and the matrix  $\hat{\mathcal{G}}^{-1}$  is defined as

$$\hat{\mathcal{G}}^{-1} \equiv \begin{pmatrix} \hat{K}_+ - 2g\rho_{TF}(x; \mu) & 0 \\ 0 & \hat{K}_- - 2g\rho_{TF}(x; \mu) \end{pmatrix} \equiv \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \quad (18)$$

Here  $\rho_{TF}(x; \mu)$  is the solution (14), and equation (17) implies that we simply added and subtracted  $2g\rho_{TF}(x; \mu)$  on the principle diagonal of the matrix operator  $\hat{G}^{-1}$ . A formal inverse of the operator  $\hat{\mathcal{G}}^{-1}$  can be found from the following equation, which defines the Green functions  $\mathcal{G}_{\pm}$ :

$$\begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix} \begin{pmatrix} \mathcal{G}_+ & 0 \\ 0 & \mathcal{G}_- \end{pmatrix} = \delta(x - x')\delta(\tau - \tau')\hat{I}.$$

Using the relation  $\ln \text{Det} = \text{Tr} \ln$ , one gets:

$$-\frac{1}{2} \ln \text{Det} \hat{G}^{-1} = -\frac{1}{2} \text{Tr} \ln \left( \hat{I} - \hat{\mathcal{G}}(\hat{\Sigma} - 2g\rho_{TF}(x; \mu)\hat{I}) \right) - \frac{1}{2} \ln \text{Det} \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix}. \quad (19)$$

The first term in Right Hand Side of (19) is free from divergencies. Let us consider the determinant of the matrix-differential operator in Right Hand Side of (19). The operators  $\mathcal{K}_\pm$  can be written in the form:

$$\mathcal{K}_\pm \equiv \pm \frac{\partial}{\partial \tau} + \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + |V(x) - \mu|. \quad (20)$$

Let us denote the eigenvalues of the operators  $\mathcal{K}_\pm$  as  $\pm i\omega_B - \lambda_n$ , where  $\omega_B$  are the bosonic Matsubara frequencies, and  $\lambda_n$  are the energy levels (which are labeled by the multi-index  $n$ ) of the operator  $-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} - |V(x) - \mu|$ . The regular part of the logarithm of the determinant in equation (19) has the sense of the free energy  $\tilde{F}_{nc}$  of the ideal gas of the over-condensate excitations:

$$\tilde{F}_{nc}(\mu) \equiv \frac{1}{2\beta} \ln \text{Det} \begin{pmatrix} \mathcal{K}_+ & 0 \\ 0 & \mathcal{K}_- \end{pmatrix} = \frac{1}{\beta} \sum_n \ln \left( 2 \sinh \frac{\beta \lambda_n}{2} \right),$$

where the regularized values of the determinants of the operators  $\mathcal{K}_\pm$  can be obtained, for instance, by means of zeta-regularization approach [27]. Then, in the leading order in  $g$ , one gets:

$$\begin{aligned} & -\frac{1}{2} \ln \text{Det} \hat{G}^{-1} \\ & \approx -\beta \tilde{F}_{nc}(\mu) + g \int_0^\beta d\tau \int dx (\mathcal{G}_+(x, \tau; x, \tau) + \mathcal{G}_-(x, \tau; x, \tau)) (\bar{\psi}_o \psi_o - \rho_{TF}(x; \mu)) \\ & \equiv -\beta F_{nc}(\mu) + 2g \int_0^\beta d\tau \int dx \rho_{nc}(x) \bar{\psi}_o \psi_o. \end{aligned} \quad (21)$$

Here  $F_{nc}$  is the free energy of the non-ideal gas of the over-condensate quasi-particles, and the last term in (21) describes a coupling of the over-condensate quasi-particles with the condensate. The density of the over-condensate quasi-particles is  $\rho_{nc}(x) \equiv -\mathcal{G}_\pm(x, \tau; x, \tau)$ , and it depends only on the spatial coordinate  $x$ . At very low temperatures and sufficiently far from the boundary of the domain occupied by the condensate, the quantity  $\rho_{nc}(x)$  can approximately be replaced by  $\rho_{nc}(0)$ , since  $\mathcal{G}_\pm(x, \tau; x, \tau)$  is almost constant over a considerable part of the condensate [29]. Eventually, one obtains

$$\begin{aligned} S_{\text{eff}}[\psi_o, \bar{\psi}_o] &= -\beta F_{nc}(\mu) + \int_0^\beta d\tau \int dx \left\{ \bar{\psi}_o(x, \tau) \left( \frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + \Lambda - V(x) \right) \psi_o(x, \tau) \right. \\ & \quad \left. - \frac{g}{2} \bar{\psi}_o(x, \tau) \bar{\psi}_o(x, \tau) \psi_o(x, \tau) \psi_o(x, \tau) \right\}, \end{aligned} \quad (22)$$

where  $\Lambda = \mu - 2g\rho_{nc}(0)$  is the renormalized chemical potential. We shall consider  $S_{\text{eff}}$  (22) as one loop effective action, where the thermal corrections over the “classical” background are taken into account. The “classical” background corresponds to the solution (14). It should be noticed that the described derivation of the effective action does not depend on spatial dimensionality, and therefore it is valid for two and three dimensions also.

In the effective action (22) it is appropriate to pass to the new variables, namely the density  $\rho(x, \tau)$  and the phase  $\varphi(x, \tau)$  of the field  $\psi_o(x, \tau)$  [21]:

$$\psi_o(x, \tau) = \sqrt{\rho(x, \tau)} e^{i\varphi(x, \tau)}, \quad \bar{\psi}_o(x, \tau) = \sqrt{\rho(x, \tau)} e^{-i\varphi(x, \tau)}. \quad (23)$$

We shall consider  $\rho(x, \tau)$  and  $\varphi(x, \tau)$  as two new independent real-valued variables of the functional integration. Now the integration measure  $\mathcal{D}\bar{\psi}_o \mathcal{D}\psi_o$  is replaced by the measure  $\mathcal{D}\rho \mathcal{D}\varphi$ . In these new variables the effective action (22) takes the form:

$$S_{\text{eff}}[\rho, \varphi] = -\beta F_{nc}(\mu) + i \int_0^\beta d\tau \int dx \left\{ \rho \partial_\tau \varphi + \frac{\hbar^2}{2m} \partial_x (\rho \partial_x \varphi) \right\} \\ + \int_0^\beta d\tau \int dx \left\{ \frac{\hbar^2}{2m} (\sqrt{\rho} \partial_x^2 \sqrt{\rho} - \rho (\partial_x \varphi)^2) + (\Lambda - V) \rho - \frac{g}{2} \rho^2 \right\}. \quad (24)$$

Here and below we denote the partial derivatives of the first order over  $\tau$  and  $x$  as  $\partial_\tau$  and  $\partial_x$ , respectively, whereas the partial derivatives of the second order — as  $\partial_\tau^2$  and  $\partial_x^2$ . Notice that equations (22) and (24) remain correct at  $V = 0$ , so that the renormalized chemical potential is still given by the equation  $\Lambda = \mu - 2g\rho_{nc}(0)$ , where  $\rho_{nc}(0)$  is the “bare” [21] density of the condensate.

## 2. The excitations spectrum

Let us consider the problem of determination of the spectrum of the low-energy quasi-particles. We shall apply the stationary phase approximation to the integral (7), where the effective action is given by (22). The corresponding stationarity point is determined from the extremum condition  $\delta(S_{\text{eff}}[\rho, \varphi]) = 0$ , which is equivalent to the couple of the Gross–Pitaevskii equations:

$$i\partial_\tau \varphi + \frac{\hbar^2}{2m} \left( \frac{1}{\sqrt{\rho}} \partial_x^2 \sqrt{\rho} - (\partial_x \varphi)^2 \right) + \Lambda - V(x) - g\rho = 0, \quad (25) \\ -i\partial_\tau \rho + \frac{\hbar^2}{m} \partial_x (\rho \partial_x \varphi) = 0.$$

Let  $\rho_0$  and  $\varphi_0$  to denote some solutions of the couple of equations (25). Substituting  $\rho_0, \varphi_0$  into the effective action (24), one obtains

$$S_{\text{eff}}[\rho_0, \varphi_0] = -\beta F_{nc}(\mu) + \frac{g}{2} \int_0^\beta d\tau \int dx \rho_0^2. \quad (26)$$

Here  $F_{nc}(\mu)$  is the free energy of the non-ideal gas of the over-condensate quasi-particles. The total free energy of the system is  $F(\mu) = -\frac{1}{\beta} S_{\text{eff}}[\rho_0, \varphi_0]$  [18].

Let us use the Thomas–Fermi approximation, which is valid at sufficiently low temperatures, and drop out the kinetic term  $(\partial_x^2 \sqrt{\rho})/\sqrt{\rho}$  in the first equation in (25). Solution with  $\partial_\tau \rho = 0 = \partial_\tau \varphi$  appears, provided the velocity field  $\mathbf{v} = m^{-1} \partial_x \varphi$  in (25) is taken equal to zero. In this case, equations (25) lead to the density of the condensate described by the solution (14), where the chemical potential  $\mu$  is replaced by  $\Lambda$ :

$$\rho_{TF}(x) \equiv \frac{\Lambda}{g} \tilde{\rho}_{TF}(x) = \frac{\Lambda}{g} \left( 1 - \frac{x^2}{R_c^2} \right) \Theta \left( 1 - \frac{x^2}{R_c^2} \right). \quad (27)$$

Explicit form of the external potential  $V(x) = \frac{m}{2} \Omega^2 x^2$  is taken into account in expression (27). The form of the solution (27) means that the quasi-condensate occupies the domain  $|x| \leq R_c$  at zero temperature. The length  $R_c$  defines the boundary of this domain,  $R_c^2 \equiv \frac{2\Lambda}{m\Omega^2}$  (in three dimensional space, this would correspond to a spherical distribution of the



condensate). In the homogeneous case given by the limit  $1/R_c \rightarrow 0$ , the Thomas–Fermi solution  $\rho_{TF}(x)$  is transformed into the density  $\rho_{TF}(0) = \Lambda/g$ , which coincides with the density of the homogeneous Bose gas [21].

Following the initial splitting (5), we suppose that the thermal fluctuations in vicinity of the stationarity point (27) are small, and therefore an analogous splitting can be written for the condensate density also:

$$\rho_0(x, \tau) = \rho_{TF}(x) + \pi_0(x, \tau). \quad (28)$$

The Gross–Pitaevskii equations (25) linearized in a vicinity of the equilibrium solution  $\rho_{TF} = \rho_{TF}(x), \varphi = \text{const}$ , then takes the form:

$$\begin{aligned} i\partial_\tau \varphi_0 - g\pi_0 + \frac{\hbar^2}{4m\rho_{TF}}\partial_x^2 \pi_0 &= 0, \\ i\partial_\tau \pi_0 - \frac{\hbar^2}{m}\partial_x(\rho_{TF}\partial_x \varphi_0) &= 0. \end{aligned} \quad (29)$$

Eliminating  $\varphi_0$  and dropping out the terms proportional to  $\hbar^4$ , we go over from (29) to the *Stringari thermal equation* [30]:

$$\frac{1}{\hbar^2 v^2} \partial_\tau^2 \pi_0 + \partial_x \left( \left( 1 - \frac{x^2}{R_c^2} \right) \partial_x \pi_0 \right) = 0, \quad (30)$$

where the parameter  $v$  has a meaning of the sound velocity in the center of the trap:

$$v^2 \equiv \frac{\rho_{TF}(0)g}{m} = \frac{\Lambda}{m}. \quad (31)$$

The substitution  $\pi_0 = e^{i\omega\tau}u(x)$  transforms (30) into the Legendre equation:

$$-\frac{\omega^2}{\hbar^2 v^2} u(x) + \frac{d}{dx} \left( \left( 1 - \frac{x^2}{R_c^2} \right) \frac{d}{dx} u(x) \right) = 0. \quad (32)$$

Since the Thomas–Fermi solution (27) is different from zero only at  $|x| \leq R_c$ , we shall consider equation (32) at  $x \in [-R_c, R_c] \subset \mathbb{R}$ , as well. After an analytical continuation  $\omega \rightarrow iE$ , equation (32) possesses the polynomial solutions, which are given by the Legendre polynomials  $P_n(x/R_c)$ , if and only if

$$\left( \frac{R_c}{\hbar v} \right)^2 E^2 \equiv \frac{2}{\hbar^2 \Omega^2} E^2 = n(n+1), \quad n \geq 0. \quad (33)$$

In other words, equation (32) leads to the spectrum of the low lying excitations:  $E_n = \hbar\Omega\sqrt{\frac{n(n+1)}{2}}$ ,  $n \geq 0$  [31]. Notice that the corresponding equation for the homogeneous Bose gas is obtained after a formal limit  $1/R_c \rightarrow 0$  in (32) at finite  $x$ . Provided the latter is still considered for the segment  $[-R_c, R_c] \ni x$  with a periodic boundary condition for  $x$ , we arrive at the discrete spectrum of the following form:  $E_k = \hbar v k$ , where  $k$  is the wave number,  $k = (\pi/R_c)n$ ,  $n \in \mathbb{Z}$ .

### §3. The two-point thermal correlation function

Let us go over to the main problem of the present paper — to the calculation of the two-point thermal correlation function of spatially non-homogeneous Bose gas described by the Hamiltonian (1):

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \equiv \langle T_\tau \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \rangle, \quad (34)$$

where  $T_\tau$  is a “ $\tau$ -chronological” ordering with respect of the imaginary time  $\tau$ , and the angular brackets  $\langle, \rangle$  correspond to averaging with respect of the Gibbs distribution [6]. We can express the correlator  $\Gamma(x_1, \tau_1; x_2, \tau_2)$  as a ratio of two functional integrals [21–24]:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) = \frac{\int e^{S[\psi, \bar{\psi}]} \bar{\psi}(x_1, \tau_1) \psi(x_2, \tau_2) \mathcal{D}\psi \mathcal{D}\bar{\psi}}{\int e^{S[\psi, \bar{\psi}]} \mathcal{D}\psi \mathcal{D}\bar{\psi}}, \quad (35)$$

where the action  $S[\psi, \bar{\psi}]$  is defined in (3).

We are interested in the behaviour of the correlators at the distances considerably smaller in comparison with the size of the whole domain occupied by the condensate. The main contribution to the behaviour of the correlation functions is due to the low lying excitations at sufficiently low temperatures [21]. To calculate  $\Gamma(x_1, \tau_1; x_2, \tau_2)$  (35), we use the method of successive functional integration first over the high-energy excitations  $\bar{\psi}_e, \psi_e$ , and then over the low-energy excitations  $\bar{\psi}_o, \psi_o$  [21, 24, 25]. We find that in the leading approximation the correlator we are interested in looks as follows [21, 25]:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\int e^{S_{\text{eff}}[\psi_o, \bar{\psi}_o]} \bar{\psi}_o(x_1, \tau_1) \psi_o(x_2, \tau_2) \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o}{\int e^{S_{\text{eff}}[\psi_o, \bar{\psi}_o]} \mathcal{D}\psi_o \mathcal{D}\bar{\psi}_o}, \quad (36)$$

where  $S_{\text{eff}}[\psi_o, \bar{\psi}_o]$  is the effective action (22). We can rewrite (36) in terms of the density–phase variables (23), and then represent the integrand in the nominator in the form of a single exponential:

$$\begin{aligned} & \Gamma(x_1, \tau_1; x_2, \tau_2) \\ & \simeq \frac{\int \exp \left( S_{\text{eff}}[\rho, \varphi] - i\varphi(x_1, \tau_1) + i\varphi(x_2, \tau_2) + \frac{1}{2} \ln \rho(x_1, \tau_1) + \frac{1}{2} \ln \rho(x_2, \tau_2) \right) \mathcal{D}\rho \mathcal{D}\varphi}{\int \exp (S_{\text{eff}}[\rho, \varphi]) \mathcal{D}\rho \mathcal{D}\varphi}. \end{aligned} \quad (37)$$

Here  $S_{\text{eff}}[\rho, \varphi]$  is the effective action (24).

Since the fluctuations of the density are suppressed at sufficiently low temperatures [28], one can replace  $\ln \rho(x_1, \tau_1)$ ,  $\ln \rho(x_2, \tau_2)$  in (37) by  $\ln \rho_{TF}(x_1)$ ,  $\ln \rho_{TF}(x_2)$ , where the density  $\rho_{TF}$  is defined by (27). In accordance with the variational principle suggested in [25], we shall estimate the functional integrals in (37) by the stationary phase method, and we shall consider only a leading approximation. For the correlation function  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ , we obtain the following estimation:

$$\begin{aligned} \Gamma(x_1, \tau_1; x_2, \tau_2) & \simeq \sqrt{\rho_{TF}(x_1) \rho_{TF}(x_2)} \\ & \times \exp \left( -S_{\text{eff}}[\rho_0, \varphi_0] + S_{\text{eff}}[\rho_1, \varphi_1] - i\varphi_1(x_1, \tau_1) + i\varphi_1(x_2, \tau_2) \right), \end{aligned} \quad (38)$$

where the variables  $\rho_0, \varphi_0$  are defined by the extremum condition  $\delta(S_{\text{eff}}[\rho, \varphi]) = 0$ , and therefore they just satisfy the stationary Gross–Pitaevskii equations (25). For  $S_{\text{eff}}[\rho_0, \varphi_0]$  we use the expression (26).

The fields  $\rho_1, \varphi_1$  are defined by the extremum condition:

$$\delta\left(S_{\text{eff}}[\rho, \varphi] - i\varphi(x_1, \tau_1) + i\varphi(x_2, \tau_2)\right) = 0. \quad (39)$$

This variational equation leads to another couple of equations of the Gross–Pitaevskii type. One of these equations turns out to be a non-homogeneous equation with the  $\delta$ -like source, while another one — a homogeneous equation. In fact, the homogeneous equation appears due to a requirement of vanishing of the coefficient at the variation  $\delta\rho(x, \tau)$ ,

$$i\partial_\tau\varphi + \frac{\hbar^2}{2m}\left(\frac{1}{\sqrt{\rho}}\partial_x^2\sqrt{\rho} - (\partial_x\varphi)^2\right) + \Lambda - V(x) - g\rho = 0. \quad (40)$$

In its turn, the non-homogeneous equation is defined by vanishing of the coefficient at the variation  $\delta\varphi(x, \tau)$ :

$$-i\partial_\tau\rho + \frac{\hbar^2}{m}\partial_x(\rho\partial_x\varphi) = i\delta(x - x_1)\delta(\tau - \tau_1) - i\delta(x - x_2)\delta(\tau - \tau_2). \quad (41)$$

Substituting the solutions  $\rho_1, \varphi_1$ , which respect (40), (41), into the effective action (24), one gets

$$S_{\text{eff}}[\rho_1, \varphi_1] = -\beta F_{nc}(\mu) - 1 + \frac{g}{2} \int_0^\beta d\tau \int dx \rho_1^2. \quad (42)$$

Further, it can consistently be assumed that the solution  $\rho_1(x, \tau)$  can be represented as a sum of  $\rho_{TF}(x)$  and of a weakly fluctuating part provided the boundary  $R_c$  is far from beginning of coordinates:  $\rho_1(x, \tau) = \rho_{TF}(x) + \pi_1(x, \tau)$  (for a comparison, see (28)). Therefore, the terms  $\sqrt{\pi_1}\partial_x^2\sqrt{\pi_1}$  and  $\partial_x\pi_1\partial_x\varphi_1$  are small and can be omitted. Taking into account a linearization near the Thomas–Fermi solution, one can finally re-write equations (40) and (41) as a couple of the following equations:

$$i\partial_\tau\varphi_1 - g\pi_1 - \frac{\hbar^2}{2m}(\partial_x\varphi_1)^2 = 0, \quad (43.1)$$

$$-i\partial_\tau\pi_1 + \frac{\hbar^2}{m}\partial_x(\rho_{TF}\partial_x\varphi_1) = i\delta(x - x_1)\delta(\tau - \tau_1) - i\delta(x - x_2)\delta(\tau - \tau_2). \quad (43.2)$$

Differentiating (43.1) over  $\tau$ , substituting the result into (43.2), and dropping out the terms of higher orders in  $g$  and  $\hbar^2$ , one obtains the following equation:

$$\frac{1}{g}\partial_\tau^2\varphi_1 + \frac{\hbar^2}{m}\partial_x(\rho_{TF}(x)\partial_x\varphi_1) = i\delta(x - x_1)\delta(\tau - \tau_1) - i\delta(x - x_2)\delta(\tau - \tau_2). \quad (44)$$

It is convenient to rewrite it as follows:

$$\frac{1}{\hbar^2 v^2}\partial_\tau^2\varphi_1 + \partial_x(\tilde{\rho}_{TF}(x)\partial_x\varphi_1) = i\frac{mg}{\hbar^2\Lambda}\left\{\delta(x - x_1)\delta(\tau - \tau_1) - \delta(x - x_2)\delta(\tau - \tau_2)\right\}, \quad (45)$$

where  $v$  means the sound velocity in the center of the trap (31), and  $\tilde{\rho}_{TF}$  is defined by (27). More specifically, solutions of equations (44), (45) depend on the coordinates of the  $\delta$ -like sources in Right Hand Side, i.e., on  $x_1, \tau_1, x_2, \tau_2$ :  $\varphi_1(x, \tau) \equiv \varphi_1(x, \tau; x_1, \tau_1, x_2, \tau_2)$ . Now,

with the help of equations (26), (42) and (43), one can calculate the contribution, which is a part in the exponent in (38):

$$\begin{aligned}
& -S_{\text{eff}}[\rho_0, \varphi_0] + S_{\text{eff}}[\rho_1, \varphi_1] \\
& \simeq \frac{g}{2} \int_0^\beta d\tau \int dx (\rho_1^2 - \rho_0^2) = \frac{g}{2} \int_0^\beta d\tau \int dx (\rho_1 - \rho_0)(\rho_1 + \rho_0) \\
& \simeq g \int_0^\beta d\tau \int dx \pi_1 \rho_0 = \int_0^\beta d\tau \int dx \left( i\partial_\tau \varphi_1 - \frac{\hbar^2}{2m} (\partial_x \varphi_1)^2 \right) \rho_0 \\
& = -\frac{\hbar^2}{2m} \int_0^\beta d\tau \int dx \rho_0 (\partial_x \varphi_1)^2 = \frac{\hbar^2}{2m} \int_0^\beta d\tau \int dx \varphi_1(x) \partial_x (\rho_0(x) \partial_x \varphi_1(x)) \\
& = \frac{i}{2} (\varphi_1(x_1, \tau_1) - \varphi_1(x_2, \tau_2)). \tag{46}
\end{aligned}$$

Substituting (46) into (38), one obtains the following approximate formula for the correlator:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\rho_{TF}(x_1) \rho_{TF}(x_2)} \exp \left( -\frac{i}{2} (\varphi_1(x_1, \tau_1) - \varphi_1(x_2, \tau_2)) \right). \tag{47}$$

It is natural to represent the solutions of equations (44), (45) in terms of the solution  $G(x, \tau; x', \tau')$  of the equation

$$\frac{1}{\hbar^2 v^2} \partial_\tau^2 G(x, \tau; x', \tau') + \partial_x \left( \left( 1 - \frac{x^2}{R_c^2} \right) \partial_x G(x, \tau; x', \tau') \right) = \frac{g}{\hbar^2 v^2} \delta(x - x') \delta(\tau - \tau'). \tag{48}$$

Bearing in mind the homogeneous equation (30), we shall call (48) as *non-homogeneous Stringari equation*. As a result, the representation (47) can be re-written as follows:

$$\begin{aligned}
\Gamma(x_1, \tau_1; x_2, \tau_2) \\
\simeq \sqrt{\rho_{TF}(x_1) \rho_{TF}(x_2)} \exp \left( -\frac{1}{2} (G(x_1, \tau_1; x_2, \tau_2) + G(x_2, \tau_2; x_1, \tau_1)) \right. \\
\left. + \frac{1}{2} G(x_1, \tau_1; x_1, \tau_1) + \frac{1}{2} G(x_2, \tau_2; x_2, \tau_2) \right). \tag{49}
\end{aligned}$$

As it is clear after [8], the function  $G(x_1, \tau_1; x_2, \tau_2)$  has a meaning of the correlation function of the phases:

$$G(x_1, \tau_1; x_2, \tau_2) = -\langle \varphi(x_1, \tau_1) \varphi(x_2, \tau_2) \rangle, \tag{50}$$

where the angle brackets in Right Hand Side should be understood as an averaging with respect to the weighted measure  $\mathcal{D}\rho \mathcal{D}\varphi \exp(S_{\text{eff}}[\rho, \varphi])$ . Substituting (50) into (49), we obtain the known approximate formula for the correlator (34) [8, 21]:

$$\langle T_\tau \hat{\psi}^\dagger(x_1, \tau_1) \hat{\psi}(x_2, \tau_2) \rangle \simeq \sqrt{\rho_{TF}(x_1) \rho_{TF}(x_2)} \exp \left( -\frac{1}{2} \langle (\varphi(x_1, \tau_1) - \varphi(x_2, \tau_2))^2 \rangle \right). \tag{51}$$

Notice that the terms in the exponent in (49) have different meanings. The terms  $G(x_1, \tau_1; x_2, \tau_2)$  and  $G(x_2, \tau_2; x_1, \tau_1)$  depend on the differences of the arguments of two-point correlation function and therefore are responsible for the behaviour of the correlator at large distances. The terms  $G(x_1, \tau_1; x_1, \tau_1)$ ,  $G(x_2, \tau_2; x_2, \tau_2)$  each depend only on a single set of the

coordinates and thus contribute to the amplitudes only. The Green function  $G(x, \tau; x', \tau')$  depends, in fact, on the difference of  $\tau$  and  $\tau'$  due to the invariance under the shifts of  $\tau$  (see. (48)). Therefore, only a dependence on the spatial coordinates remains provided the corresponding thermal arguments coincide. Then, it is possible to represent the correlation function as follows:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)} \exp\left(-\frac{1}{2}(G(x_1, \tau_1; x_2, \tau_2) + G(x_2, \tau_2; x_1, \tau_1))\right), \quad (52)$$

where  $\tilde{\rho}(x_1), \tilde{\rho}(x_2)$  are the renormalized densities [18]. The solution  $G(x_1, \tau_1; x_2, \tau_2)$  of equation (48) is defined up to a purely imaginary additive constant, which has a meaning of a global phase. As it is seen from (51), this constant does not influence the fluctuations.

## §4. The asymptotics of the correlation function

Therefore, the problem concerning the study of the asymptotical behaviour of the two-point thermal correlation function  $\Gamma(x_1, \tau_1; x_2, \tau_2)$  (34), which is given by the representation (37), is reduced to solution of the non-homogeneous Stringari equation (48). The corresponding answer (or its asymptotics) should be subsequently substituted into (52). In the present section, we shall obtain explicitly solutions of equation (48), and we shall consider the corresponding representations for the asymptotics of  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ . Let us begin with the limiting case of equation (48), which corresponds to a homogeneous Bose gas.

### 1. The homogeneous Bose gas

In this subsection we shall consider the asymptotical behaviour of the correlation function of the homogeneous Bose gas. As it was mentioned above, the homogeneous case corresponds to  $V(x) \equiv 0$ , and the related equation appears from (48) at  $1/R_c \rightarrow 0$ :

$$\frac{1}{\hbar^2 v^2} \partial_\tau^2 G(x, \tau; x', \tau') + \partial_x^2 G(x, \tau; x', \tau') = \frac{g}{\hbar^2 v^2} \delta(x - x') \delta(\tau - \tau'). \quad (53)$$

We consider (53) for the domain  $[-R_c, R_c] \times [0, \beta] \ni (x, \tau)$  with the periodic boundary conditions for each variable (in other words, we consider (53) on the torus  $S^1 \times S^1 \ni (x, \tau)$ ). The  $\delta$ -functions in Right Hand Side of (53) are treated as the corresponding periodic  $\delta$ -functions. This allows us to represent the solution of this equation as the formal double Fourier series:

$$\begin{aligned} G(x, \tau; x', \tau') &= \left(\frac{-g}{\hbar^2 v^2}\right) (2\beta R_c)^{-1} \sum_{\omega, k} \frac{e^{i\omega(\tau-\tau') + ik(x-x')}}{\omega^2/(\hbar^2 v^2) + k^2} \\ &= \left(\frac{-g}{2\beta R_c}\right) \sum_{\omega, k} \frac{e^{i\omega(\tau-\tau') + ik(x-x')}}{\omega^2 + E_k^2}, \end{aligned} \quad (54)$$

where  $\omega = (2\pi/\beta)l, l \in \mathbb{Z}$ . The notation for the energy  $E_k = \hbar v k$ , where  $k = (\pi/R_c)n, n \in \mathbb{Z}$ , is used in the representation (54). Besides, the representation (54) requires a regularization, which consists in neglect of the term given by  $\omega = k = 0$ .

Using (54), one can deduce two important asymptotical representations for the Green function. In the limit of zero temperature and of infinite size of the domain occupied by the

Bose gas, one can go over to the asymptotics of  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ . When a strong inequality  $\beta^{-1} \equiv k_B T \gg \hbar v / R_c$  is valid, we obtain:

$$G(x, \tau; x', \tau') \simeq \frac{g}{2\pi\hbar v} \ln \left\{ 2 \left| \sinh \frac{\pi}{\hbar\beta v} (|x - x'| + i\hbar v(\tau - \tau')) \right| \right\} - \frac{g}{4\beta R_c} \frac{|x - x'|^2}{\hbar^2 v^2} + \mathcal{C}, \quad (55)$$

where  $|x - x'| \leq 2R_c$ ,  $|\tau - \tau'| \leq \beta$ , and  $\mathcal{C}$  is some constant, which is not written explicitly. When an opposite inequality  $\beta^{-1} \equiv k_B T \ll \hbar v / R_c$  is valid, we obtain:

$$G(x, \tau; x', \tau') \simeq \frac{g}{2\pi\hbar v} \ln \left\{ 2 \left| \sinh \frac{i\pi}{2R_c} (|x - x'| + i\hbar v(\tau - \tau')) \right| \right\} - \frac{g}{4\beta R_c} |\tau - \tau'|^2 + \mathcal{C}', \quad (56)$$

where  $|x - x'| \leq 2R_c$ ,  $|\tau - \tau'| \leq \beta$ , and  $\mathcal{C}'$  is another constant.

Let us substitute the estimate (55) into the representation (52) and take simultaneously the limit  $\beta\hbar v / R_c \rightarrow 0$  (the size is growing faster than inverse temperature). Then, we obtain the following expression for the correlator in question:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)} \left| \sinh \frac{\pi}{\hbar\beta v} (|x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)) \right|^{-g/2\pi\hbar v}. \quad (57)$$

Further, applying the relation (56) and taking the limit  $R_c/(\beta\hbar v) \rightarrow 0$  (the inverse temperature grows faster than the size), we obtain for  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ :

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)} \left| \sinh \frac{i\pi}{2R_c} (|x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)) \right|^{-g/2\pi\hbar v}. \quad (58)$$

It follows from (57) and (58), that in the limit of zero temperature,  $(\hbar\beta v)^{-1} \rightarrow 0$ , and of infinite size,  $1/R_c \rightarrow 0$ , the two-point correlation function behaves like

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)}}{|x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)|^{1/\theta}}. \quad (59)$$

The latter formula is valid in the limit  $\beta\hbar v / R_c \rightarrow 0$ , as well as in the limit  $R_c/(\beta\hbar v) \rightarrow 0$ . In (59),  $\theta$  denotes the critical exponent:  $\theta \equiv 2\pi\hbar v / g$ , and the arguments  $x_1$  and  $x_2$ ,  $\tau_1$  and  $\tau_2$ , are assumed to be sufficiently close each to other. Using the notations  $v = \sqrt{\Lambda/m}$  for the sound velocity and  $\rho = \Lambda/g$  for the density of the homogeneous ideal Bose gas, we obtain for the critical exponent the following universal expression [11, 12]:

$$\theta = \frac{2\pi\hbar\rho}{mv}. \quad (60)$$

## 2. The trapped Bose gas. High temperature case: $k_B T \gg \hbar v / R_c$

Let us consider the case of non-homogeneous Bose gas, which is described by the Hamiltonian (1) with the external potential  $V(x) \equiv \frac{m}{2}\Omega^2 x^2$ . In the present subsection we are mainly following the content of the paper [20]. Let us consider the non-homogeneous Stringari equation (48) and write it again, for convenience:

$$\frac{1}{\hbar^2 v^2} \partial_\tau^2 G(x, \tau; x', \tau') + \partial_x \left( \left( 1 - \frac{x^2}{R_c^2} \right) \partial_x G(x, \tau; x', \tau') \right) = \frac{g}{\hbar^2 v^2} \delta(x - x') \delta(\tau - \tau'). \quad (61)$$

We consider (61) for the arguments  $(x, \tau) \in [-R_c, R_c] \times [0, \beta]$  with the periodic boundary condition only with respect to  $\tau$  (contrary to equation (53),  $\delta(x - x')$  is a usual Dirac's  $\delta$ -function with a support at the point  $x' \in \mathbb{R}$ ). The Green function satisfying (61) can be written as a formal Fourier series:

$$G(x, \tau; x', \tau') = \frac{1}{\beta} \sum_{\omega} e^{i\omega(\tau - \tau')} G_{\omega}(x, x'), \quad (62)$$

where  $\omega = (2\pi/\beta)l$ ,  $l \in \mathbb{Z}$ . The spectral density  $G_{\omega}(x, x')$  in (62) is then governed by the equation

$$-\frac{\omega^2}{\hbar^2 v^2} G_{\omega}(x, x') + \frac{d}{dx} \left( \left( 1 - \frac{x^2}{R_c^2} \right) \frac{d}{dx} G_{\omega}(x, x') \right) = \frac{g}{\hbar^2 v^2} \delta(x - x'). \quad (63)$$

Solution of equation (63) can be obtained in terms of the Legendre functions of the first and second kind,  $P_{\nu}(x/R_c)$  and  $Q_{\nu}(x/R_c)$ , which are linearly independent solutions of the homogeneous Legendre equation (32). As a result, we get:

$$G_{\omega}(x, x') = \frac{gR_c}{2\hbar^2 v^2} \epsilon(x - x') \left\{ Q_{\nu} \left( \frac{x}{R_c} \right) P_{\nu} \left( \frac{x'}{R_c} \right) - Q_{\nu} \left( \frac{x'}{R_c} \right) P_{\nu} \left( \frac{x}{R_c} \right) \right\}, \quad (64)$$

where  $\nu$  looks as follows:

$$\nu = -\frac{1}{2} + \sqrt{\frac{1}{4} - \left( \frac{R_c}{\hbar v} \right)^2 \omega^2},$$

and  $\epsilon(x - x')$  is the sign function  $\epsilon(x) \equiv \text{sign}(x)$ . Validity of the solution (64) can be verified by direct substitution of (64) into (63), where the following expression for the Wronskian of two linearly independent solutions  $P_{\nu}$  and  $Q_{\nu}$  [32] should be used:

$$P_{\nu}(y) \frac{d}{dy} Q_{\nu}(y) - Q_{\nu}(y) \frac{d}{dy} P_{\nu}(y) = (1 - y^2)^{-1},$$

and the rule of differentiation of the sign function is:  $(d/dx) \epsilon(x) = 2\delta(x)$ .

Provided a dependence on  $\tau$  is neglected, an equation, which arises as a result of calculation of the correlation function accordingly to [25], looks analogously to equation (61), but a factor  $\beta^{-1}$  is present in its Right Hand Side instead of  $\delta(\tau - \tau')$ . In this case, the corresponding solution of the non-homogeneous equation (i. e., the fundamental solution)  $G(x; x')$  takes the form

$$G(x; x') = \frac{1}{\beta} G_0(x, x'), \quad (65.1)$$

where

$$\begin{aligned} G_0(x, x') &= \frac{gR_c}{2\hbar^2 v^2} \epsilon(x - x') \left\{ Q_0 \left( \frac{x}{R_c} \right) - Q_0 \left( \frac{x'}{R_c} \right) \right\} \\ &= \frac{gR_c}{(2\hbar v)^2} \ln \left[ \frac{\left( 1 + \frac{|x-x'|}{2R_c} \right)^2 - \frac{(x+x')^2}{4R_c^2}}{\left( 1 - \frac{|x-x'|}{2R_c} \right)^2 - \frac{(x+x')^2}{4R_c^2}} \right]. \end{aligned} \quad (65.2)$$

An explicit form for the simplest Legendre functions  $P_0(x) = 1$  and  $Q_0(x) = \frac{1}{2} \ln \frac{1+x}{1-x}$  is essential for obtaining the relations (65). The fundamental solution  $G(x; x')$  (65) becomes

equal to zero when its arguments coincide. A substitution of (65) into the representation (49) gives the following result for the stationary correlation function [17, 18, 33, 34]:

$$\Gamma(x_1; x_2) \simeq \sqrt{\rho_{TF}(x_1)\rho_{TF}(x_2)} \exp\left(-\frac{gR_c}{\beta(2\hbar v)^2} \ln \left[ \frac{1 + \frac{|x_1-x_2|}{R_c} - \frac{x_1x_2}{R_c^2}}{1 - \frac{|x_1-x_2|}{R_c} - \frac{x_1x_2}{R_c^2}} \right] \right). \quad (66)$$

Before studying the behaviour of the correlation function, which depends on  $\tau$ , it should be noticed that solutions of the homogeneous Legendre equation (32) can be added to the Green function (62), since the latter respects the non-homogeneous equation. In order to ensure a correct asymptotical behaviour of the spectral density we are interested in at large  $|\omega|$ , we are free to add such a term at  $|\omega| \neq 0$  and obtain the following expression:

$$G_\omega(x, x') = \frac{gR_c}{2\hbar^2 v^2} \epsilon(x - x') \left\{ Q_\nu\left(\frac{x}{R_c}\right) P_\nu\left(\frac{x'}{R_c}\right) - Q_\nu\left(\frac{x'}{R_c}\right) P_\nu\left(\frac{x}{R_c}\right) \right\} \\ - i \frac{gR_c}{2\hbar^2 v^2} \left\{ \frac{2}{\pi} Q_\nu\left(\frac{x}{R_c}\right) Q_\nu\left(\frac{x'}{R_c}\right) + \frac{\pi}{2} P_\nu\left(\frac{x'}{R_c}\right) P_\nu\left(\frac{x}{R_c}\right) \right\}. \quad (67.1)$$

The Green function given by (62) and (67.1) can be represented in the form, which allows to study the corresponding asymptotical behaviour. In the case of strong inequality  $\beta^{-1} = k_B T \gg \hbar v / R_c$ , we approximately obtain for non-zero frequencies:  $|\omega| \gg \hbar v / (2R_c)$ . Let us take into account the following asymptotics of the Legendre functions [32, 36]:

$$\left\{ \begin{matrix} P_\nu \\ Q_\nu \end{matrix} \right\} (\cos \theta) = \frac{\Gamma(\nu + 1)}{\Gamma(\nu + 3/2)} \left(\frac{2}{\pi}\right)^{\delta/2} \frac{1}{\sin^{1/2} \theta} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \left[ \left(\nu + \frac{1}{2}\right) \theta + \frac{\pi}{4} \right] + O(\nu^{-1}) \\ \approx \left(\frac{2}{\pi}\right)^{\delta/2} \frac{1}{(\nu \sin \theta)^{1/2}} \left\{ \begin{matrix} \sin \\ \cos \end{matrix} \right\} \left[ \left(\nu + \frac{1}{2}\right) \theta + \frac{\pi}{4} \right], \quad (67.2)$$

where  $\varepsilon < \theta < \pi - \varepsilon$ ,  $\varepsilon > 0$ ,  $|\arg \nu| < \pi/2$ , and the notation  $\cos \theta \equiv x/R_c$  is adopted. Only the up or the down expressions must be simultaneously chosen inside the curly brackets  $\{\dots\}$  in (67.2). Here  $\delta = 1$  corresponds to  $P_\nu$ , and  $\delta = -1$  corresponds to  $Q_\nu$ . Substituting (67.2) into (67.1), we determine the behaviour of  $G_\omega(x, x')$  at large  $|\omega|$ :

$$G_\omega(x, x') \simeq -\frac{g}{2\hbar v |\omega|} \frac{1}{\sqrt{\sin \theta \sin \theta'}} \exp\left(-\frac{R_c}{\hbar v} |\omega| |\theta - \theta'|\right). \quad (68)$$

When the coordinates  $x_1, x_2$  are chosen to be far from the boundary of the trap,  $x_1, x_2 \ll R_c$ , but at the same time the inequalities  $|x_1 - x_2| \ll \frac{x_1 + x_2}{2}$  and  $|x_1 - x_2| \ll R_c$  are valid, the corresponding limit should be called as *quasi-homogeneous*. In this case, the function  $G_0(x, x')$  (65.2) can be approximated up to a second order (including the latter) as follows:

$$G_0(x, x') \simeq \frac{\Lambda}{2\hbar^2 v^2 \rho_{TF}(S)} |x - x'|. \quad (69)$$

Here  $S$  means a half-sum of the spatial arguments of the correlator,  $S \equiv \frac{x_1 + x_2}{2}$ , and  $v$  is the sound velocity (31). In the quasi-homogeneous limit, equation (68) can be re-written in the following form:

$$G_\omega(x, x') \simeq -\frac{\Lambda}{2\hbar v \rho_{TF}(S)} \frac{\exp(-(\hbar v)^{-1} |\omega| |x - x'|)}{|\omega|}. \quad (70)$$



Substitution of (69) and (70) into the series (62) leads to the answer for the Green function (i.e., for the correlator of the phases) (50):

$$G(x, \tau; x', \tau') \simeq \frac{\Lambda}{2\pi\hbar v \rho_{TF}(S)} \ln \left\{ 2 \left| \sinh \frac{\pi}{\hbar\beta v} (|x - x'| + i\hbar v(\tau - \tau')) \right| \right\}. \quad (71)$$

Therefore, the Green function (52) takes the following form at  $\beta^{-1} \gg \hbar v/R_c$ :

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)}}{|\sinh \frac{\pi}{\hbar\beta v} (|x_1 - x_2| + i\hbar v(\tau_1 - \tau_2))|^{1/\theta(S)}}, \quad (72)$$

where the critical exponent  $\theta(S)$  depends now only on the half-sum of the coordinates  $S$ :

$$\theta(S) = \frac{2\pi\hbar\rho_{TF}(S)}{mv}. \quad (73)$$

The result (72), which is valid for the spatially non-homogeneous case, is in a correspondence with the estimation (57) obtained above for the homogeneous Bose gas. Therefore, the estimation (72) is also concerned with validity of the appropriate condition that the size of the domain occupied by the Bose condensate grows faster than inverse temperature, i.e., with the condition  $\hbar\beta v/R_c \rightarrow 0$ .

The relation (72) can be simplified for two important limiting cases. Provided the condition

$$1 \ll \frac{|x_1 - x_2|}{\hbar\beta v} \ll \frac{R_c}{\hbar\beta v} \quad (74)$$

is fulfilled in the quasi-homogeneous case, we obtain from (72) that the correlation function decays exponentially:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)} \exp\left(-\frac{1}{\xi(S)}||x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)|\right), \quad (75)$$

$$\xi^{-1}(S) = \frac{\Lambda}{2\beta\hbar^2 v^2 \rho_{TF}(S)}.$$

The correlation length  $\xi(S)$  is defined by the relation (75), which depends now on the half-sum of the coordinates:

$$\xi(S) \equiv \frac{\hbar\beta v}{\pi} \theta(S) = \frac{2\hbar^2 \beta \rho_{TF}(S)}{m}. \quad (76)$$

In an opposite limit,

$$\frac{|x_1 - x_2|}{\hbar\beta v}, \frac{|\tau_1 - \tau_2|}{\beta} \ll 1 \ll \frac{R_c}{\hbar\beta v}, \quad (77)$$

the asymptotics of  $\Gamma(x_1, \tau_1; x_2, \tau_2)$  takes the following form:

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)}}{||x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)|^{1/\theta(S)}}. \quad (78)$$

The obtained asymptotics (78) is analogous to the estimation (59), which characterizes the spatially homogeneous Bose gas. But the critical exponent  $\theta(S)$  (73) differs from  $\theta$  (60), since it depends on the spatial coordinates.

### 3. The trapped Bose gas. Low temperature case: $k_B T \ll \hbar v / R_c$

Let us pass to another case which also admits investigation of the asymptotical behaviour of the two-point correlator  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ . As in the previous subsection, we begin with the non-homogeneous Stringari equation (48), which can be written in the following form:

$$\begin{aligned} \partial_\tau^2 G(x, \tau; x', \tau') + \frac{1}{\alpha^2} \partial_{(x/R_c)} \left( \left( 1 - \frac{x^2}{R_c^2} \right) \partial_{(x/R_c)} G(x, \tau; x', \tau') \right) \\ = \frac{g}{R_c} \delta \left( \frac{x - x'}{R_c} \right) \delta(\tau - \tau'), \end{aligned} \quad (79)$$

where the notation  $\alpha \equiv R_c / (\hbar v)$  is introduced. The asymptotical behaviour can be investigated in two cases, and these cases may be characterized in terms of  $\alpha$ :  $\beta/\alpha \ll 1$  (the previous subsection) and  $\beta/\alpha \gg 1$  (see below). The functions

$$\sqrt{n + \frac{1}{2}} P_n \left( \frac{x}{R_c} \right), \quad n \geq 0,$$

where  $P_n(x/R_c)$  are the Legendre polynomials, constitute a complete orthonormal system in the space  $L_2[-R_c, R_c]$ . This fact allows to obtain the following representation for the Green function  $G(x, \tau; x', \tau')$  in the form of the generalized double Fourier series:

$$G(x, \tau; x', \tau') = \left( \frac{-g}{\beta R_c} \right) \sum_{\omega} \sum_{n=0}^{\infty} \frac{n + 1/2}{\omega^2 + E_n^2} P_n \left( \frac{x}{R_c} \right) P_n \left( \frac{x'}{R_c} \right) e^{i\omega(\tau - \tau')}. \quad (80)$$

In (80), as well as in (54), (62), the notation  $\sum_{\omega}$  denotes a sum over the Bose frequencies  $\omega = (2\pi/\beta)l$ ,  $l \in \mathbb{Z}$ , and the following notation for the energy levels (33) is adopted:

$$E_n = \hbar \Omega \sqrt{\frac{n(n+1)}{2}} = \frac{\sqrt{n(n+1)}}{\alpha}. \quad (81)$$

Let us note that a transition from (54) to (80) takes place provided the wave functions and the dispersion relation are appropriately substituted.

After summation over the frequencies and after regularization consisting in neglecting of the term corresponding to zero values of  $\omega$  and  $n$ ,  $G(x, \tau; x', \tau')$  (80) takes the form:

$$\begin{aligned} G(x, \tau; x', \tau') \\ = \left( \frac{-g}{\beta R_c} \right) \left[ \left( \frac{\beta}{2\pi} \right)^2 \sum_{l=1}^{\infty} \frac{\cos \left( \frac{2\pi \Delta \tau}{\beta} l \right)}{l^2} + \frac{\beta}{2} \sum_{n=1}^{\infty} \frac{n + 1/2}{E_n} P_n \left( \frac{x}{R_c} \right) P_n \left( \frac{x'}{R_c} \right) \right. \\ \left. \times \left( \coth \left( \frac{\beta}{2} E_n \right) \cosh(E_n \Delta \tau) - \sinh(E_n \Delta \tau) \right) \right], \end{aligned} \quad (82)$$

where  $\Delta \tau \equiv |\tau - \tau'|$ . The obtained representation (82) admits an investigation for two limiting cases:  $\beta/\alpha \ll 1$  (this case agrees with the estimation (75) of the previous subsection) and  $\beta/\alpha \gg 1$ .

Indeed, putting  $\tau = \tau'$  in the case  $\beta/\alpha \ll 1$ , we obtain [35] the following relation:

$$\begin{aligned} G(x, \tau; x', \tau) &= -\frac{g\beta}{24R_c} - \frac{g}{\beta R_c} \sum_{n=1}^{\infty} \frac{n + 1/2}{E_n^2} P_n \left( \frac{x}{R_c} \right) P_n \left( \frac{x'}{R_c} \right) \\ &= \frac{gR_c}{2\beta \hbar^2 v^2} \ln \left( 1 + \frac{|x - x'|}{R_c} - \frac{xx'}{R_c^2} \right) + \frac{gR_c}{\beta \hbar^2 v^2} \left( \frac{1}{2} - \ln 2 - \frac{\beta^2}{24\alpha^2} \right). \end{aligned} \quad (83)$$

Let us observe that Right Hand Side of (83) respects the non-homogeneous equation

$$\frac{1}{\alpha^2} \partial_{(x/R_c)} \left( \left( 1 - \frac{x^2}{R_c^2} \right) \partial_{(x/R_c)} G(x, \tau; x', \tau) \right) = \frac{g}{\beta R_c} \left( \delta \left( \frac{x - x'}{R_c} \right) - \frac{1}{2} \right). \quad (84.1)$$

Provided the spatial arguments in the relation (83) are equated, one can obtain the following equality:

$$\begin{aligned} & \frac{G(x, \tau; x, \tau) + G(x', \tau; x', \tau)}{2} \\ &= \frac{g\alpha^2}{4\beta R_c} \ln \left[ \left( 1 - \frac{x^2}{R_c^2} \right) \left( 1 - \frac{(x')^2}{R_c^2} \right) \right] + \frac{gR_c}{\beta \hbar^2 v^2} \left( \frac{1}{2} - \ln 2 - \frac{\beta^2}{24\alpha^2} \right). \end{aligned} \quad (84.2)$$

Direct substitution demonstrates that (84.2) satisfies an equation, where Left Hand Side is the same as in (84.1), while only the constant term  $\frac{-g}{2\beta R_c}$  is present in Right Hand Side. Therefore, a subtraction of Right Hand Side of (84.2) from Right Hand Side of (83) results exactly in the fundamental solution  $G(x; x')$  (65). The Green function  $G(x; x')$  becomes equal to zero at  $x = x'$ , and it respects an equation of the type of (84.1) but with  $\frac{g}{\beta} \delta(x - x')$  in Right Hand Side.

With the help of the series (83), we may bring the exponent in (49) into the form:

$$\left( \frac{-g}{2\beta R_c} \right) \sum_{n=1}^{\infty} \frac{n+1/2}{E_n^2} \left( P_n \left( \frac{x}{R_c} \right) - P_n \left( \frac{x'}{R_c} \right) \right)^2. \quad (85)$$

As it is seen from (83) and (84.2), the relation (85) is nothing else but the fundamental solution  $G(x; x')$  (65) taken with an opposite sign. Thus, the representation (83), being substituted into (49), leads to the formula for the “equal-time” correlator  $\Gamma(x_1, \tau; x_2, \tau)$ , which has the same form as Right Hand Side of (66). Taking into account the quasi-homogeneity conditions, which were discussed in the previous subsection, and recalling the corresponding estimation (69), we obtain for  $G(x; x')$ :

$$G(x, x') \simeq \frac{\Lambda}{2\beta \hbar^2 v^2 \rho_{TF}(S)} |x - x'|.$$

In its turn, the latter formula leads to the estimation:

$$\Gamma(x_1, \tau; x_2, \tau) \simeq \sqrt{\rho_{TF}(x_1) \rho_{TF}(x_2)} \exp \left( -\frac{1}{\xi(S)} |x_1 - x_2| \right), \quad (86)$$

where  $\xi(S)$  is the correlation length (76). As a result, equations (75) and (86) demonstrate an agreement of the estimations based on two different representations for the Green function  $G(x, \tau; x', \tau')$ : the first one is in the form of the trigonometric Fourier series (62) (where either (64) or (67) is used to express  $G_\omega(x, x')$ ), and the second one is in the form of the series (82), which runs over the principle quantum numbers (obtained from the generalized double Fourier series (80)).

Now let us turn to the case  $\beta/\alpha \gg 1$ , where  $\beta E_n \gg 1$ ,  $\forall n = 1, 2, \dots$ . In other words, let us suppose that  $k_B T \ll E_n$  and, so,  $k_B T \ll \hbar \Omega$ . Then, from (82) one obtains:

$$\begin{aligned} G(x, \tau; x', \tau') &= \frac{-g\beta}{4R_c} \left[ \left( \frac{1}{2} - \frac{\Delta\tau}{\beta} \right)^2 - \frac{1}{12} \right] \\ &- \frac{g}{2\hbar v} \sum_{n=1}^{\infty} \frac{n+1/2}{\sqrt{n(n+1)}} P_n \left( \frac{x}{R_c} \right) P_n \left( \frac{x'}{R_c} \right) \exp \left( -\sqrt{n(n+1)} \frac{\Delta\tau}{\alpha} \right). \end{aligned} \quad (87)$$

Let us note that a difference between two neighbouring energy levels (81) can be estimated. After some appropriate series expansions, which are valid at  $n > 1$ , one obtains:

$$\begin{aligned} E_{n+1} - E_n &\approx \frac{1}{\alpha} \left[ 1 + \frac{1}{8} \frac{1}{(n+1)^2} + \frac{7}{128} \frac{1}{(n+1)^4} \cdots \right] \\ &\approx \frac{1}{\alpha} \left[ 1 + \frac{1}{8n^2} - \frac{1}{4n^3} + \frac{55}{128n^4} \cdots \right]. \end{aligned} \quad (88)$$

Right Hand Side of (88) demonstrates that the levels (81) can approximately be treated as equi-distant provided the inverse powers of  $n$  are neglected in (88), say, for the values  $n > n_0 = 10$ . In its turn, the following series expansion is valid:

$$\frac{n+1/2}{\sqrt{n(n+1)}} = 1 + \frac{1}{8n^2} - \frac{1}{8n^3} + \frac{15}{128n^4} - \cdots \quad (89)$$

It is remarkable that the terms  $\sim n^{-1}$  are absent both in (88) and (89). Let us remind that leading asymptotical estimations, which are obtainable with so-called *logarithmic accuracy*, are important for physical applications. As it will be clear below, the problem at hands just admits an estimation with leading logarithmic accuracy. In this case, the inverse powers of  $n$  can be omitted with the same accuracy in (88) and (89) at  $n > n_0 = 10$ . For such approximation, the energy levels (81) turn out to be equi-distant, while the corresponding ratio  $\frac{n+1/2}{\sqrt{n(n+1)}}$  in (87) becomes equal to unity. Convergency of the series (87) is not affected in this situation, while the term omitted can be estimated.

Let us consider the exponent in (87):

$$\sqrt{n(n+1)} \frac{\Delta\tau}{\alpha} = \frac{\Delta\tau}{\alpha} \left( n + \frac{1}{2} \right) - \frac{\Delta\tau}{\alpha} \left( \frac{1}{8n} - \frac{1}{16n^2} + \frac{5}{128n^3} - \cdots \right). \quad (90)$$

The second term in (90) can also be neglected in the exponent at  $n > n_0$ , provided  $\Delta\tau/\alpha \ll 1$ . This means that the series entering into (87) can approximately be written as follows:

$$\begin{aligned} \sum_{n=1}^{n_0} \frac{n+1/2}{\sqrt{n(n+1)}} P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right) \exp\left(-\sqrt{n(n+1)} \frac{\Delta\tau}{\alpha}\right) \\ + e^{-\Delta\tau/(2\alpha)} \sum_{n=n_0+1}^{\infty} P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right) \left(e^{-\Delta\tau/\alpha}\right)^n, \end{aligned} \quad (91)$$

where  $n_0$  is the number, which is fixed (its specific value is forbidden to go to infinity). The correction, say  $\tilde{c}$ , omitted in the representation (91) can be estimated:

$$\begin{aligned} |\tilde{c}| &\leq \left( \left( 1 + \frac{1}{n_0^2} \right) \left( 1 + \frac{\Delta\tau \text{ const}}{\alpha n_0} \right) - 1 \right) \\ &\times e^{-\Delta\tau/(2\alpha)} \sum_{n=n_0+1}^{\infty} \left| P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right) \right| \left( e^{-\Delta\tau/\alpha} \right)^n. \end{aligned}$$

It can also be demonstrated that absolute value of the first term in (91) does not exceed

$2n_0$ . Using (91), let us put (87) in the following form:

$$\begin{aligned}
G(x, \tau; x', \tau') &\approx \frac{-g\beta}{4R_c} \left[ \left( \frac{1}{2} - \frac{\Delta\tau}{\beta} \right)^2 - \frac{1}{12} \right] \\
&- \frac{g}{2\hbar v} \sum_{n=1}^{n_0} \left[ \frac{n+1/2}{\sqrt{n(n+1)}} \exp\left(-\sqrt{n(n+1)} \frac{\Delta\tau}{\alpha}\right) - \exp\left(-\left(n+\frac{1}{2}\right) \frac{\Delta\tau}{\alpha}\right) \right] \\
&\quad \times P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right) \\
&- \frac{g}{2\hbar v} e^{-\Delta\tau/(2\alpha)} \left[ \sum_{n=0}^{\infty} t^n P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right) - 1 \right],
\end{aligned} \tag{92}$$

where  $t \equiv \exp(-\Delta\tau/\alpha)$ . As it can be seen from (80), the equation obtained (92) is valid for the case when  $\tau$  and  $\tau'$  are close either to zero or to  $\beta$ , as well as for the case when only  $\tau$  or  $\tau'$  is close to  $\beta$ . Besides, we assume that  $\tau \neq \tau'$  in order to keep convergency of (92).

Using the known series [35]

$$\sum_{n=0}^{\infty} t^n P_n(\cos \vartheta_1) P_n(\cos \vartheta_2) = \frac{4}{\pi} \frac{1}{u_+ u_-} \mathbb{K}(\kappa), \quad 0 < t < 1, \tag{93}$$

$$\begin{aligned}
u_+ &\equiv \sqrt{1 - 2t \cos(\vartheta_1 + \vartheta_2) + t^2}, \quad u_- \equiv \sqrt{1 - 2t \cos(\vartheta_1 - \vartheta_2) + t^2}, \\
\kappa &= \frac{u_+ - u_-}{u_+ + u_-},
\end{aligned} \tag{94}$$

where  $\mathbb{K}$  is a complete elliptic integral of the first kind, one can estimate the approximate representation for the Green function (92). Indeed, let us put  $\cos \vartheta_1 \equiv x/R_c \ll 1$  and  $\cos \vartheta_2 \equiv x'/R_c \ll 1$ . Then the following estimations for  $u_+$  and  $u_-$  can be obtained:

$$u_+ \approx 1 + t - \frac{t}{1+t} \frac{(x+x')^2}{2R_c^2} \approx 2 - \frac{\Delta\tau}{\alpha} - \frac{(x+x')^2}{4R_c^2}, \tag{95.1}$$

$$u_- \approx \left( (1-t)^2 + t \frac{(x-x')^2}{R_c^2} \right)^{1/2} \approx \frac{||x-x'| + i\hbar v \Delta\tau|}{R_c} \equiv u_*, \tag{95.2}$$

where it is assumed that  $\Delta\tau/\alpha \ll 1$  and

$$\frac{t}{1+t} \approx \frac{1}{2} \left( 1 - \frac{\Delta\tau}{2\alpha} \right).$$

Provided the terms of second order smallness are neglected, an estimation  $\kappa \simeq 1 - u_*$  arises for  $\kappa$  (94), where  $u_*$  implies the corresponding approximate value of  $u_-$  given by Right Hand Side of (95.2). When  $\kappa \sim 1$ , several leading terms of the asymptotical expansion of the function  $\mathbb{K}(\kappa)$  [35] can be written down:

$$\mathbb{K}(\kappa) \approx \mathbb{K}(1 - u_*) \approx \frac{u_*}{4} \left( \left( \frac{2}{u_*} + 1 \right) \ln \frac{8}{u_*} - 1 \right), \quad u_* \ll 1. \tag{96}$$

The estimation presented (96) does not contain the terms  $\sim (u_*)^2$ , and the other terms which contain higher powers of  $u_*$  are absent. This is due to the fact that under the imposed condition of quasi-homogeneity, the value  $u_*$  (95.2) is treated as a quantity of first order smallness. Therefore, the corresponding value of the argument  $\kappa$ , i. e.,  $\kappa \simeq 1 - u_*$ , is written neglecting the contributions of second order smallness.

Let us point out that the first and the second terms in  $G(x, \tau; x', \tau')$  (92) are not small, because the number  $n_0$  can occur to be large. Besides, the inequality  $\beta/\alpha \gg 1$  implies that  $g\beta/R_c \gg g/(\hbar v)$ . However, provided a smallness of the quantities  $x/R_c$ ,  $x'/R_c$  and  $\Delta\tau/\alpha$  is taken into account, it is seen that the first two terms in (92) are less important in comparison to the third one, which can be logarithmically large at sufficiently small  $u_*$ . It is why, we neglect the first two terms and write down the leading contribution to the Green function (92) with the logarithmic accuracy:

$$\begin{aligned} G(x, \tau; x', \tau') & \simeq \left( \frac{-g}{4\pi\hbar v} \right) \left( 1 + \frac{2R_c}{||x - x'| + i\hbar v(\tau - \tau')|} \right) \ln \frac{8R_c}{||x - x'| + i\hbar v(\tau - \tau')|} \\ & \approx \left( \frac{-g}{2\pi\hbar v} \right) \frac{1}{u_*} \ln \frac{8}{u_*}. \end{aligned} \quad (97)$$

Here it is assumed that  $u_*$  is given by (95.2), and the following conditions of validity of the logarithmic estimation are respected:

$$n_0 \lesssim \frac{1}{u_*}, \quad 1 \ll \frac{1}{u_*} \ll \frac{1}{u_*} \ln \frac{1}{u_*}. \quad (98)$$

Substitution of (97) into (52) gives us the following estimation for the two-point correlator  $\Gamma(x_1, \tau_1; x_2, \tau_2)$ :

$$\Gamma(x_1, \tau_1; x_2, \tau_2) \simeq \frac{\sqrt{\tilde{\rho}(x_1)\tilde{\rho}(x_2)}}{||x_1 - x_2| + i\hbar v(\tau_1 - \tau_2)|^{1/\bar{\theta}}}. \quad (99)$$

In (99), the notation  $\bar{\theta}$  for the critical exponent is introduced:

$$\bar{\theta} \equiv \frac{2\pi\hbar v}{g} u_*. \quad (100)$$

The critical exponent  $\bar{\theta}$  depends on  $u_*$  (95.2), and therefore it is a function of differences of the coordinates. Apart from the inequalities (98), the following estimations can be obtained to characterize the relations (99), (100):

$$\begin{aligned} \frac{1}{\hbar v} & \ll \frac{R_c}{\hbar v||x - x'| + i\hbar v(\tau - \tau')|} \ll \frac{\beta}{||x - x'| + i\hbar v(\tau - \tau')|}, \\ \frac{1}{\hbar v} & \ll \frac{\beta}{R_c} \ll \frac{\beta}{||x - x'| + i\hbar v(\tau - \tau')|}. \end{aligned}$$

The estimation obtained (99) (together with the critical exponent  $\bar{\theta}$  (100)) constitutes the main result of the present subsection devoted to the case given by  $k_B T \ll \hbar v/R_c$ . From a comparison with the spatially homogeneous Bose gas, one can see that the derivation of the estimate (99) is just analogous to a transition from the relations (56), (58) to the final asymptotics (59). Then, validity of the corresponding limit  $R_c/(\hbar\beta v) \rightarrow 0$  means that the result (99) is also concerned with the fact that the condensate boundary  $R_c$  increases slower than the inverse temperature.

Let us note that due to (98), a specific value of  $n_0$  can be related to the size of the trap  $R_c$ : at a fixed range of deviations between the spatial coordinates  $x$  and  $x'$ , increasing of  $R_c$  leads to an increasing of an upper bound for admissible values of  $n_0$ . However, due to (98), the estimation obtained for  $G(x, \tau; x', \tau')$  (97) does not depend explicitly on a specific

choice of the number  $n_0$ . On the other hand, at sufficiently large values of  $n$ , the following asymptotics for the Legendre polynomials  $P_n$  is valid [35]:

$$P_n(\cos \vartheta) = \sqrt{\frac{2}{\pi n \sin \vartheta}} \cos \left[ \left( n + \frac{1}{2} \right) \vartheta - \frac{\pi}{4} \right] + O(n^{-3/2}), \quad 0 < \vartheta < \pi. \quad (101)$$

Let us assume that the number  $n_0$  is large enough to substitute (101) into the series

$$\sum_{n=1}^{\infty} \left( e^{-\Delta\tau/\alpha} \right)^n P_n\left(\frac{x}{R_c}\right) P_n\left(\frac{x'}{R_c}\right),$$

which is a part of the representation (92) (see also (91)), in order to obtain its estimation with the logarithmic accuracy. Eventually, the following result takes place for  $G(x, \tau; x', \tau')$  (92):

$$G(x, \tau; x', \tau') \simeq \left( \frac{-g}{2\pi\hbar v} \right) \left( 1 + \frac{S^2}{2R_c^2} \right) \ln \frac{R_c}{||x - x'| + i\hbar v(\tau - \tau')|}, \quad (102)$$

where  $S = \frac{x_1 + x_2}{2}$ . In the limit  $1/R_c \rightarrow 0$ , the total coefficient in front of the logarithm in (102) takes the value  $-1/\theta$ , where the critical exponent  $\theta$  is defined like in (59), (60).

The usage of the asymptotics (101) implies that the eigen-functions are approximated by the base consisting of the plane waves, which correspond to an almost homogeneous Bose gas. Therefore, a replacement of the asymptotics (97) by the asymptotics (102) can be explained as a transition, at increasing of  $R_c$ , to smaller scales characterized by small ratios  $x/R_c$ ,  $x'/R_c$  and  $|x - x'|/R_c$  (the quasi-homogeneity condition). Then, the result (99) together with the critical exponent (100) demonstrate an effect of finiteness of the size of the domain occupied by the spatially non-homogeneous Bose gas. This follows just from the employment of the Legendre polynomials as the base of one-particle states.

## §5. Conclusion

The model considered in this paper describes a spatially non-homogeneous one-dimensional Bose gas with a weak repulsive coupling placed into an external harmonic potential. The paper deals with an application of the functional integration approach to the calculation of the two-point thermal correlation function of the non-homogeneous Bose gas. The temperatures that are low enough for the quasi-condensate to be created in the Bose system in question (see §2) are studied. The functional integral representation for the two-point correlation function in question is estimated by means of the stationary phase approximation in the way proposed in [25]. The main results are obtained for the case when the size of the domain occupied by the quasi-condensate increases, while the temperature of the system goes to zero. It is demonstrated that the behaviour of the correlation function near zero temperature has a power-like dependence, and it is governed by the critical exponent. In contrast with the case of spatial homogeneity of the Bose gas, the presence of the external potential is manifested in the non-homogeneity of the critical exponent. The latter turns out to become a function of the same spatial and thermal arguments as the correlator itself. The dependence of the critical exponent on these spatial arguments is in correspondence with the limiting behaviour of the ratio of the size of the trap to the inverse temperature provided both of them are increasing.

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